Perspectives on Nonlinearity in Quantum Mechanics

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It is with great pleasure that I dedicate this contribution to my friend and collaborator, Prof. Dr. Heinz-Dietrich Doebner, on the special occasion of his retirement from the Arnold Sommerfeld Institute for Mathematical Physics.

Abstract

Earlier H.-D. Doebner and I proposed a family of nonlinear time-evolution equations for quantum mechanics associated with certain unitary representations of the group of diffeomorphisms of physical space. Such nonlinear Schrödinger equations may describe irreversible, dissipative quantum systems. We subsequently introduced the group of nonlinear gauge transformations necessary to understand the resulting quantum theory, deriving and interpreting gauge-invariant parameters that characterize (at least partially) the physical content. Here I first review these and related results, including the coupled nonlinear Schrödinger-Maxwell theory, for which I also introduce the gauge-invariant (hydrodynamical) equations of motion. Then I propose a further, radical generalization. An enlarged group $\mathcal{G}$ of nonlinear transformations, modeled on the general linear group $GL(2, \mathbf{R})$, leads to a beautiful, apparently unremarked symmetry between the wave function's phase and the logarithm of its amplitude. The equations Doebner and I proposed are embedded in a wider, natural family of nonlinear time-evolution equations, invariant (as a family) under $\mathcal{G}$. Furthermore there exist $\mathcal{G}$-invariant quantities that reduce to the usual expressions for probability density and flux for linearizable quantum theories in a particular gauge. Thus $\mathcal{G}$ may be interpreted as generalizing further our notion of nonlinear gauge transformation.

1 Families of Nonlinear Schrödinger Equations

About nine years ago, H.-D. Doebner and I introduced a certain family of nonlinear Schrödinger equations. We were led to these equations not by any prior inclination to study nonlinear quantum mechanics, but by our desire to interpret quantum-mechanically a class of representations of an infinite-dimensional, nonrelativistic current algebra, and the corresponding group [1, 2, 3]. We proposed these equations as candidates for describing quantum systems with dissipation.

To review the development briefly, we sought self-adjoint representations of the infinite-dimensional Lie algebra of densities and currents, given at arbitrary time t by

$$[\,\rho_{op}(f_1),\,\rho_{op}(f_2)]\,=\,0\,,\quad [\,\rho_{op}(f),\,J_{op}(\mathbf{g})]\,=\,i\hbar\rho_{op}(\mathbf{g}\cdot\nabla f)\,,$$
$$[J_{op}(\mathbf{g}_1),\,J_{op}(\mathbf{g}_2)]\,=\,-i\hbar J_{op}([\mathbf{g}_1,\,\mathbf{g}_2])\,, \tag{1}$$

where the f's are real-valued C^∞ functions on the physical space $\mathbf{R}^n$, the $\mathbf{g}$'s are C^∞ vector fields on $\mathbf{R}^n$, and $[\,\mathbf{g}_1,\mathbf{g}_2\,]\,=\,\mathbf{g}_1\cdot\nabla\mathbf{g}_2-\mathbf{g}_2\cdot\nabla\mathbf{g}_1$ is the usual Lie bracket [4, 5, 6, 7]. The N-particle Bose or Fermi representations of (1) may be written

$$\rho^N_{op}(f)\,\psi^{(s,a)}(\mathbf{x}_1,\ldots\mathbf{x}_N)\;=\;m\sum_{j=1}^{N}f(\mathbf{x}_j)\psi^{(s,a)}(\mathbf{x}_1,\ldots\mathbf{x}_N),$$
$$J^N_{op}(\mathbf{g})\,\psi^{(s,a)}(\mathbf{x}_1,\ldots\mathbf{x}_N)\;=\;\frac{\hbar}{2i}\sum_{j=1}^{N}\{\,\mathbf{g}(\mathbf{x}_j)\cdot\nabla_j\psi^{(s,a)}(\mathbf{x}_1,\ldots\mathbf{x}_N)$$
$$+\;\nabla_j\cdot[\,\mathbf{g}(\mathbf{x}_j)\psi^{(s,a)}(\mathbf{x}_1,\ldots\mathbf{x}_N)\,]\,\}\,, \tag{2}$$

where the $\psi^{(s,a)}$ are (respectively) symmetric or antisymmetric square-integrable functions of the N particle coordinate variables. There exists a family of related but unitarily inequivalent representations of (1), parameterized by the real number D, leading to physically distinct quantizations [8, 9]:

$$J^{N,D}_{op}(\mathbf{g})\;=\;J^N_{op}(\mathbf{g})\;+\;D\,\rho^N_{op}(\nabla\cdot\mathbf{g}). \tag{3}$$

Here D is a constant with the dimensions of a diffusion coefficient. Even in the case of one-particle quantum mechanics, interpreting these representations posed a challenge.

In the usual notation for operator-valued distributions, write (suppressing the superscripts) $\rho_{op}(f)=\int_X\rho_{op}(\mathbf{x})f(\mathbf{x})d\mathbf{x}$ and $J_{op}(\mathbf{g})=\int_X\mathbf{J}_{op}(\mathbf{x})\cdot\mathbf{g}(\mathbf{x})d\mathbf{x}$. Then, for a single particle at time t, take the expectation values $m\,\rho\,(\mathbf{x},t)\;=\;\langle\psi_t|\,\rho_{op}(\mathbf{x})\,|\psi_t\rangle$ and $m\,\mathbf{j}\,(\mathbf{x},t)\;=\;\langle\psi_t|\,\mathbf{J}_{op}(\mathbf{x})\,|\psi_t\rangle$. When $D=0$ the usual expressions are recovered for the probability density and flux in the Schrödinger representation:

$$\rho\,=\,\overline{\psi}\psi\,,\quad \mathbf{j}\,=\,\frac{\hbar}{2mi}\,[\,\overline{\psi}\nabla\psi-(\nabla\overline{\psi})\psi\,]\,. \tag{4}$$

For arbitrary D, one obtains instead $\mathbf{j}^D\;=\;\mathbf{j}\;-\;D\,\nabla(\overline{\psi}\psi)$. Imposing the equation of continuity $\partial_t\,\rho\;=\;-\nabla\cdot\mathbf{j}^D$ then gives, as a kinematical constraint on the time-evolution of ψ, a Fokker-Planck type of equation: $\partial_t\,\rho\;=\;-\nabla\cdot\mathbf{j}\;+\;D\,\nabla^2\rho\,$.

No linear time-evolution equation for ψ obeys this constraint. Rather we derived an interesting family of nonlinear Schrödinger equations, with the purely imaginary functional $i\hbar(D/2)\,\nabla^2\rho/\rho$ multiplying ψ on the right-hand side. That is, this particular form of nonlinearity was forced on us by the current algebra representation. And without linearity as an axiom, we also could not eliminate *a priori* the possibility of additional, real nonlinear functionals multiplying ψ. Doebner and I restricted these to homogeneous rational expressions with no more than two derivatives in the numerator. Defining (for convenience) $\hat{\mathbf{j}} = (m/\hbar)\,\mathbf{j} = (1/2i)\,[\,\overline{\psi}\nabla\psi - (\nabla\overline{\psi})\psi\,]$, we introduced the real, homogeneous functionals $R_1[\psi], \ldots, R_5[\psi]$ given by

$$R_1 \;=\; \frac{\nabla\cdot\hat{\mathbf{j}}}{\rho}\,, \quad R_2 \;=\; \frac{\nabla^2\rho}{\rho}\,, \quad R_3 \;=\; \frac{\hat{\mathbf{j}}^{\,2}}{\rho^2}\,, \quad R_4 \;=\; \frac{\hat{\mathbf{j}}\cdot\nabla\rho}{\rho^2}\,, \quad R_5 \;=\; \frac{(\nabla\rho)^2}{\rho^2}\,. \tag{5}$$

The family of nonlinear Schrödinger equations became then:

$$i\hbar\frac{\partial\psi}{\partial t} \;=\; H_0\,\psi \;+\; \frac{i}{2}\hbar D\,R_2[\psi]\,\psi \;+\; \hbar D'\sum_{j=1}^{5} c_j\,R_j[\psi]\,\psi\,, \tag{6}$$

where D' is another diffusion coefficient, the c_j are real and dimensionless, and

$$H_0\,\psi \;=\; \frac{1}{2m}\,[-i\hbar\nabla - (e/c)\mathbf{A}(\mathbf{x},t)]^2\,\psi \;+\; [\,V \;+\; e\,\Phi(\mathbf{x},t)]\,\psi\,. \tag{7}$$

Below we shall see how an important subclass of (6), and certain more general nonlinear Schrödinger equations, can be obtained from the linear Schro‘ödinger equation via nonlinear gauge transformations. Eq. (6) contains as special cases a remarkable variety of nonlinear modifications of quantum mechanics proposed independently by other researchers [10, 11, 12, 13, 14, 15, 16], though without our fundamental motivation for the nonlinearity and typically without the above local, pure imaginary nonlinear functional multiplying ψ.

Using the expansion $\nabla^2\,\psi/\psi \;=\; iR_1[\psi] + (1/2)\,R_2[\psi] - R_3[\psi] - (1/4)\,R_5[\psi]$, let us rewrite this family of equations as in Ref. [20], with some additional terms:

$$\begin{aligned} i\,\frac{\dot\psi}{\psi} \;\;&=\;\; i\left[\,\sum_{j=1}^{2}\nu_j R_j[\psi] \;+\; \frac{\nabla\cdot(\mathcal{A}(\mathbf{x},t)\rho)}{\rho}\,\right] \;+ \\ &\left[\,\sum_{j=1}^{5}\mu_j R_j[\psi] \;+\; U(\mathbf{x},t) \;+\; \frac{\nabla\cdot(\mathcal{A}_1(\mathbf{x},t)\rho)}{\rho} \;+\; \frac{\mathcal{A}_2(\mathbf{x},t)\cdot\hat{\mathbf{j}}}{\rho} \;+\; \alpha_1\,\ln\rho \;+\; \alpha_2\,S\,\right]. \end{aligned} \tag{8}$$

Here S is the phase of ψ, U is a (sufficiently smooth) external, real-valued, time-dependent scalar function; and $\mathcal{A}$, $\mathcal{A}_1$, and $\mathcal{A}_2$ are distinct (sufficiently smooth) external, real-valued, time-dependent vector fields. Eq. (6) is obtained from Eq. (8) with the following substitutions:

$$\nu_1 = -\frac{\hbar}{2m}\,, \quad \nu_2 = \frac{1}{2}D\,, \quad \mathcal{A} = \frac{e}{2mc}\mathbf{A}\,,$$

$$\mu_1 = D'c_1\,, \quad \mu_2 = -\frac{\hbar}{4m} + D'c_2\,, \quad \mu_3 = \frac{\hbar}{2m} + D'c_3\,, \quad \mu_4 = D'c_4\,, \quad \mu_5 = \frac{\hbar}{8m} + D'c_5\,,$$

$$U(\mathbf{x},t) = \frac{1}{\hbar}\,[\,V(\mathbf{x},t)\,+\,e\,\Phi\,]\,+\,\frac{e^2}{2m\hbar c^2}\,\mathbf{A}^2, \quad \mathcal{A}_1 = 0\,, \quad \mathcal{A}_2 = -\frac{e}{mc}\mathbf{A}\,,$$

$$\alpha_1 \,=\, \alpha_2 \,=\, 0. \tag{9}$$

The coefficients ν_j $(j = 1,2)$, μ_j $(j = 1,\ldots,5)$, and α_j $(j = 1,2)$ are taken to be continuously differentiable, real-valued functions of t. The motivation for this expansion, the reason behind the introduction of terms with α_1, α_2, and $\mathcal{A}_1 \neq 0$, and the reason for permitting the coefficients to be time-dependent, all stem from the discussion of nonlinear gauge transformations in the next section.

Finally, let us introduce here a further, natural generalization of Eq. (8). Let us insert into the imaginary part of the right-hand side the terms $\nu_3 R_3$, $\nu_4 R_4$, and $\nu_5 R_5$, as well as new external scalar and vector fields, to achieve full symmetry between the real and imaginary parts [17]. Thus we have, in effect, allowed for *complexification* of all the coefficients and external fields. The equation becomes:

$$i\,\frac{\dot\psi}{\psi} \,=\, i\left[\,\sum_{j=1}^{5}\nu_j R_j[\psi]\,+\,\mathcal{T}(\mathbf{x},t)\,+\,\frac{\nabla\cdot(\mathcal{A}(\mathbf{x},t)\rho)}{\rho}\,+\,\frac{\mathcal{D}(\mathbf{x},t)\cdot\hat{\mathbf{j}}}{\rho}\,+\,\delta_1\,\ln\rho\,+\,\delta_2\,S\right]\,+$$
$$\left[\,\sum_{j=1}^{5}\mu_j R_j[\psi]\,+\,U(\mathbf{x},t)\,+\,\frac{\nabla\cdot(\mathcal{A}_1(\mathbf{x},t)\rho)}{\rho}\,+\,\frac{\mathcal{A}_2(\mathbf{x},t)\cdot\hat{\mathbf{j}}}{\rho}\,+\,\alpha_1\,\ln\rho\,+\,\alpha_2\,S\right], \tag{10}$$

where $\mathcal{T}$ is a new external scalar field, and $\mathcal{D}$ a new external vector field. Note that the heat equation and other interesting equations of mathematical physics fall within this family. Some equations with soliton-like solutions are also included [18]. But the equation of continuity relating ρ and $\mathbf{j}^D$ no longer holds. Evidently when $\nu_3 = \nu_4 = \nu_5 = \delta_1 = \delta_2 = 0$, $\mathcal{T} = 0$, and $\mathcal{D} = 0$, we recover Eq. (8). When the remaining values are as in Eq. (9) with $D = D' = 0$, we are back with the linear Schrödinger equation.

We shall see that the generalization of Eq. (8) to Eq. (10) follows from a further, natural extension of the notion of nonlinear gauge transformation.

2 Time-Dependent Nonlinear Gauge Transformations

Let us write $\psi = R\exp[\,iS\,]$, where the amplitude R and the phase S are real. Then $\rho = R^2$ and $\mathbf{j} = (\hbar/m)\,R^2\,\nabla S$. While R is gauge invariant, S is not: under the usual, unitary gauge transformations of quantum mechanics, $R' = R$ but $S' = S + \theta(\mathbf{x},t)$. Then $\rho' = \rho$, while $\mathbf{j}' = \mathbf{j} + (\hbar/m)\,R^2\,\nabla\theta$.

If we begin with the linear Schrödinger equation in the absence of a vector potential, i.e., $i\hbar\partial_t \psi \,=\, -(\hbar^2/2m)\,\nabla^2\psi \,+\, V\,\psi$, then the transformed wave function $\psi' \,=\, R'\exp[\,iS'\,]$ satisfies $i\hbar\partial_t\psi' \,=\, (\hbar^2/2m)\,[-i\nabla \,-\, \mathrm{grad}\,\theta]^2\,\psi' \,+\, [V \,-\, \hbar\dot{\theta}]\,\psi'$. This observation can actually motivate introduction of the external electromagnetic gauge potentials $\mathbf{A}$ and Φ, and the "minimally coupled" Schrödinger equation whose Hamiltonian is given by Eq. (7). When we begin with (7), we have that ψ' satisfies the transformed equation obtained by substituting the gauge-transformed potentials: $\mathbf{A}' \,=\, \mathbf{A} \,+\, (\hbar c/e)\,\mathrm{grad}\,\theta$ and $\Phi' \,=\, \Phi \,-\, (\hbar/e)\,\dot{\theta}$. A gauge-invariant current can now be written $\mathbf{J}^{\mathrm{gi}} \,=\, \mathbf{j} - (e/mc)\,\rho\,\mathbf{A}$, with $\partial_t\,\rho \,=\, -\nabla\cdot\mathbf{J}^{\mathrm{gi}}$. The physical fields $\mathbf{B} \,=\, \nabla\times\mathbf{A}$ and $\mathbf{E} \,=\, -\nabla\Phi - (1/c)\,\partial_t\,\mathbf{A}$ are likewise gauge invariant. All this is elementary, and standard. It sets the pattern for consideration of nonlinear gauge transformations for nonlinear Schrödinger equations.

In the latter context we (necessarily) abandon the usual, tacit assumption that gauge transformations act linearly and unitarily. Doebner and I introduced a group of nonlinear transformations leaving our class of equations invariant as a family [19, 20],

$$R' \,=\, R\,, \quad S' \,=\, \Lambda S + \gamma\ln R + \theta\,, \tag{11}$$

where in general γ and Λ are continuously differentiable, real-valued functions of t, $\Lambda \neq 0$, and θ is a continuously differentiable, real-valued function of $\mathbf{x}$ and t. Then $(\Lambda_1,\,\gamma_1,\,\theta_1)\;(\Lambda_2,\,\gamma_2,\,\theta_2) \,=\, (\Lambda_1\Lambda_2,\,\gamma_1 \,+\, \Lambda_1\gamma_2,\,\theta_1 \,+\, \Lambda_1\theta_2)$. The original justification for taking these to be gauge transformations was the argument, put forth by many theorists, that any physical quantum-mechanical measurement could be reduced to a sequence of positional measurements at different times; with the system subjected to external force fields between measurements [21, 22]. Under Eq. (11),

$$\rho' \,=\, \overline{\psi'}\,\psi' \,=\, \rho\,,$$

$$\hat{\mathbf{j}}' \,=\, \frac{1}{2i}\,[\,\overline{\psi'}\,\nabla\psi' - (\nabla\overline{\psi'}\,)\,\psi'\,] \,=\, \Lambda\,\hat{\mathbf{j}} \,+\, \frac{\gamma}{2}\nabla\rho \,+\, \rho\nabla\theta\,. \tag{12}$$

Keeping the interpretation of $\rho \,=\, |\psi|^2$ as the positional probability density, and writing invariant force fields in terms of the external potentials, the outcomes of all measurements do remain invariant. Eq. (11) also has other nice properties: it is strictly local, and it respects a certain separation condition for (many-particle) product wave functions [23, 24]. If ψ obeys a Schrödinger equation of the type in Eq. (8), then ψ' transformed by (11) obeys another equation in the family, with transformed coefficients and external fields. The coefficients are given by:

$$\nu_1' = \frac{\nu_1}{\Lambda}\,, \quad \nu_2' = -\frac{\gamma}{2\Lambda}\nu_1 + \nu_2\,,$$

$$\mu_1' = -\frac{\gamma}{\Lambda}\nu_1 + \mu_1\,, \quad \mu_2' = \frac{\gamma^2}{2\Lambda}\nu_1 - \gamma\nu_2 - \frac{\gamma}{2}\mu_1 + \Lambda\mu_2\,,$$

$$\mu_3' = \frac{\mu_3}{\Lambda}\,, \quad \mu_4' = -\frac{\gamma}{\Lambda}\mu_3 + \mu_4\,, \quad \mu_5' = \frac{\gamma^2}{4\Lambda}\mu_3 - \frac{\gamma}{2}\mu_4 + \Lambda\mu_5\,,$$

$$\alpha_1' \,=\, \Lambda\alpha_1 \,-\, \frac{\gamma}{2}\,\alpha_2 \,+\, \frac{1}{2}\left(\frac{\dot\Lambda}{\Lambda}\gamma \,-\, \dot\gamma\right), \quad \alpha_2' \,=\, \alpha_2 \,-\, \frac{\dot\Lambda}{\Lambda}\,, \tag{13}$$

while the transformed vector and scalar fields are

$$\begin{aligned} \mathcal{A}' \,&=\, \mathcal{A} \,-\, \frac{\nu_1}{\Lambda}\,\nabla\theta\,, \\ \mathcal{A}_1' \,&=\, \Lambda\,\mathcal{A}_1 \,-\, \gamma\,\mathcal{A} \,-\, \frac{\gamma}{2}\,\mathcal{A}_2 \,+\, \left(\frac{\gamma}{\Lambda}\,\nu_1 - \mu_1 + \frac{\gamma}{\Lambda}\,\mu_3 - \mu_4\right)\nabla\theta\,, \\ \mathcal{A}_2' \,&=\, \mathcal{A}_2 - \frac{2\mu_3}{\Lambda}\,\nabla\theta\,, \\ U' \,&=\, \Lambda\,U \,-\, \dot\theta \,+\, \left(\frac{\dot\Lambda}{\Lambda} - \alpha_2\right)\theta \,+\, \frac{\mu_3}{\Lambda}\,[\,\nabla\theta\,]^{\,2} + \\ &\quad \left(\mu_4 - \mu_3\,\frac{\gamma}{\Lambda}\right)\nabla^2\theta \,+\, \frac{\gamma}{2}\,\nabla\cdot\mathcal{A}_2 \,-\, \mathcal{A}_2\cdot\nabla\theta. \end{aligned} \tag{14}$$

Regarding Eqs. (13), note how the time-dependence of γ and Λ in Eq. (11) *requires* that the ν_j, μ_j , and α_j in Eq. (8) be time-dependent, and that the α_j be allowed nonzero values. The terms with α_1 and α_2 were, respectively, first introduced by Bialynicki-Birula and Micielski [25] and by Kostin [26]. Likewise, we see in (14) how the $\mathcal{A}_1$ and $\mathcal{A}_2$ terms in Eq. (8) are needed. Nonlinear Schrödinger equations with arbitrary values of $\mathcal{A}_2$ were considered by Haag and Bannier [27], while as far as I know the field $\mathcal{A}_1$ was first considered in Ref. [20]. An important subclass of Eq. (8) is linearizable by means of nonlinear gauge transformations; for this subclass, the physics is unchanged from ordinary quantum mechanics.

The coefficients, the external fields, and many of the nonlinear functionals in Eq. (8) are not gauge invariant. But we do have a current $\mathbf{J}^{\rm gi}$, invariant under nonlinear gauge transformations, that enters the continuity equation $\dot\rho = -\nabla\cdot\mathbf{J}^{\rm gi}$, given by

$$\mathbf{J}^{\rm gi} \,=\, -\,2\nu_1\,\hat{\mathbf{j}} \,-\, 2\nu_2\nabla\rho \,-\, 2\rho\mathcal{A}\,. \tag{15}$$

This reduces, of course, to the usual gauge-invariant current in the linear case [20]. Now, the existence of $\mathbf{J}^{\rm gi}$ means that our earlier assumption about all measurements being reducible to a succession of positional measurements is unnecessarily restrictive. It is sufficient that all measurements be expressible in terms of gauge-invariant quantities; and we have available for this the density ρ , the current $\mathbf{J}^{\rm gi}$, and gauge-invariant force fields (see below).

Doebner and I also introduced gauge-invariant parameters:

$$\tau_1 = \nu_2 - \frac{1}{2}\mu_1\,, \quad \tau_2 = \nu_1\mu_2 - \nu_2\mu_1\,, \quad \tau_3 = \frac{\mu_3}{\nu_1}\,, \quad \tau_4 = \mu_4 - \mu_1\frac{\mu_3}{\nu_1}\,,$$

$$\tau_5 = \nu_1\mu_5 - \nu_2\mu_4 + \nu_2^2\,\frac{\mu_3}{\nu_1}\,,$$

$$\beta_1 \,=\, \nu_1\,\alpha_1 \,-\, \nu_2\,\alpha_2 \,+\, \nu_2\,\frac{\dot\nu_1}{\nu_1} - \dot\nu_2\,, \quad \beta_2 \,=\, \alpha_2 \,-\, \frac{\dot\nu_1}{\nu_1}\,. \tag{16}$$

Some discussion of the physics behind these parameters may found in Ref. [19]; in particular, $\tau_1 \neq 0$, $\tau_4 \neq 0$, or $\beta_2 \neq 0$ violates time-reversal invariance; $\tau_3 \neq -1$ or $\tau_4 \neq 0$ breaks Galileian invariance; and in all these cases τ_2 corresponds to the *observed* value of $\hbar^2/8m^2$ (no longer can we identify the gauge-dependent quantity $-\nu_1$ with the gauge-independent, observable constant $\hbar/2m$). Thus the classical limit can be taken in a gauge-invariant manner by letting $\tau_2 \to 0$.

Let me also remark here that the gauge-invariant parameter β_2 is naturally interpreted as a coefficient of friction, as it contributes (see below) a term $-\beta_2\,(\mathbf{J}^{\mathrm{gi}}/\rho)$ to the expression for $\partial_t\,(\mathbf{J}^{\mathrm{gi}}/\rho)$.

Continuing the discussion in Ref. [20] we have also gauge-invariant fields. Set

$$\hat U \,=\, -\,\nu_1\,U \,-\, \tau_3\,\mathcal{A}^2 \,-\, (\tau_4 - 2\tau_1\tau_3)\,\nabla\cdot\mathcal{A} \,+\, \mathcal{A}\cdot\mathcal{A}_2 \,-\, \nu_2\,\nabla\cdot\mathcal{A}_2\,, \tag{17}$$

so that under nonlinear gauge transformation,

$$\hat U\,' \,=\, \hat U \,+\, \frac{\nu_1}{\Lambda}\,\dot\theta \,+\, \frac{\nu_1}{\Lambda}\,\alpha_2\,\theta \,-\, \nu_1\,\frac{\dot\Lambda}{\Lambda^2}\,\theta\,. \tag{18}$$

Eq. (17) corrects algebraic errors in Ref. [20]. The field $\hat U$ is easily reduced to $(1/2m)\,(V \,+\, e\,\Phi)$ for the linear Schrödinger equation. We have the new gauge-invariant vector fields,

$$\begin{aligned} \mathcal{A}_1^{gi} \,&=\, \nu_1\,\mathcal{A}_1 \,+\, \left(\frac{2\nu_2\mu_3}{\nu_1} - \mu_1 - \mu_4\right)\mathcal{A} \,-\, \nu_2\,\mathcal{A}_2\,, \\ \mathcal{A}_2^{gi} \,&=\, \frac{\nu_1}{2\mu_3}\,\mathcal{A}_2 - \mathcal{A}\,, \end{aligned} \tag{19}$$

as well as magnetic and (generalized) electric plus other potential force fields,

$$\begin{aligned} \mathcal{B} = \nabla\times\mathcal{A} \,&=\, \frac{e}{2mc}\,\mathbf{B}, \\ \mathcal{E} \,=\, -\,\nabla\hat U - \frac{\partial\mathcal{A}}{\partial t} \,-\, \beta_2\,\mathcal{A} \,&=\, -\frac{1}{2m}\nabla V \,+\, \frac{e}{2m}\,\mathbf{E}. \end{aligned} \tag{20}$$

Thus $\hat U \,=\, (1/2m)(V + e\,\Phi)$ in general, and $\mathbf{E} \,=\, -\nabla\Phi \,-\, (1/c)\,\partial_t\mathbf{A} \,-\, (\beta_2/c)\,\mathbf{A}$. Notice the extra term associated with Kostin's nonlinearity; without it, $\mathcal{E}$ is not gauge invariant. This leads in turn to an interesting modification of one of Maxwell's equations:

$$\nabla\times\mathbf{E} \,=\, -\,\frac{1}{c}\,\frac{\partial\,\mathbf{B}}{\partial t} \,-\, \frac{\beta_2}{c}\,\mathbf{B}\,. \tag{21}$$

3 Gauge-Invariant Equations of Motion

Using the (hydrodynamical) variables ρ and $\mathbf{V} = \mathbf{J}^{\mathrm{gi}}/\rho$, it is straightforward to write down in manifestly gauge-invariant form the equations of motion corresponding to Eq. (8). We have in all cases the useful relation $\nabla \times \mathbf{V} = -2\mathcal{B} = (e/mc)\mathbf{B}$, and the continuity equation $\partial_t \rho = -\nabla \cdot \mathbf{J}^{\mathrm{gi}}$. In addition,

$$
\begin{aligned}
\frac{\partial}{\partial t}\left(\frac{\mathbf{J}^{\mathrm{gi}}}{\rho}\right) = {} & \nabla\left[2\tau_1 \nabla \cdot \left(\frac{\mathbf{J}^{\mathrm{gi}}}{\rho}\right) + 2\tau_2 \frac{\nabla^2 \rho}{\rho} + \frac{1}{2}\tau_3 \left(\frac{\mathbf{J}^{\mathrm{gi}}}{\rho}\right)^2\right] \\
& + \nabla\left[\left(2\tau_1 \left[1+\tau_3\right] - \tau_4\right)\left(\frac{\mathbf{J}^{\mathrm{gi}}}{\rho}\right) \cdot \frac{\nabla \rho}{\rho} + 2\tau_5 \frac{(\nabla\rho)^2}{\rho^2}\right] \\
& + \nabla\left[2\frac{\nabla \cdot (\mathcal{A}_1^{gi}\rho)}{\rho} - 2\tau_3 \mathcal{A}_2^{gi} \cdot \left(\frac{\mathbf{J}^{\mathrm{gi}}}{\rho}\right) + 2\beta_1 \ln \rho\right] \\
& - \beta_2 \left(\frac{\mathbf{J}^{\mathrm{gi}}}{\rho}\right) - \frac{1}{m}\nabla V + \frac{e}{m}\mathbf{E}.
\end{aligned} \tag{22}
$$

Now we have the expected values of position, velocity, and acceleration:

$$
\begin{aligned}
< \mathbf{x} > &= \int \mathbf{x}\, \rho\,(\mathbf{x})\, d\mathbf{x}\,, \\
< \mathbf{v} > &= \frac{\partial < \mathbf{x} >}{\partial t} = \int \rho \left(\frac{\mathbf{J}^{\mathrm{gi}}}{\rho}\right) d\mathbf{x}\,, \\
< \mathbf{a} > &= \frac{\partial < \mathbf{v} >}{\partial t} = \int \rho \left[\frac{1}{2}\nabla\left(\frac{\mathbf{J}^{\mathrm{gi}}}{\rho}\right)^2 + \left(\frac{\mathbf{J}^{\mathrm{gi}}}{\rho}\right) \times \frac{e}{mc}\mathbf{B} + \frac{\partial}{\partial t}\left(\frac{\mathbf{J}^{\mathrm{gi}}}{\rho}\right)\right] d\mathbf{x}\,.
\end{aligned} \tag{23}
$$

Note that in Eqs. (22)-(23), the force laws governing interaction with the external electric and magnetic fields are unchanged from linear quantum mechanics.

4 The Enlarged Gauge Group

To this point, the amplitude R and the phase S have a fundamentally different status, both in linear quantum mechanics and in our nonlinear variations: R is gauge invariant, and physically observable; while S is not. This asymmetry seems more and more puzzling as one comes to appreciate the flexibility of description offered by nonlinear

quantum time-evolutions, allowing for instance linear quantum mechanics to be written in a nonlinear gauge. Why should we be required to combine the *gauge* field S with the *physical* field R into a single complex-valued function ψ, and then through the Schrödinger equation couple both R and S to the gauge potentials? Why not instead try to couple gauge-dependent quantitites to each other, and correspondingly, physical fields to each other?

In addition, we remark that just as the formula (15) for the gauge-invariant current $\mathbf{J}^{\mathrm{gi}}$ depended on two coefficients and one external potential in the nonlinear time-evolution equation (8), there is no *a priori* principle that forbids the formula for the gauge-invariant *probability density* from likewise depending on coefficients and external potentials in the time-evolution equation. This is important as we consider enlarging the nonlinear gauge group further.

To achieve the desired generalization, define $T = \ln R$, so that $\ln \psi = T + iS$, and consider the transformations

$$\begin{pmatrix} S' \\ T' \end{pmatrix} = \begin{pmatrix} \Lambda & \gamma \\ \lambda & \kappa \end{pmatrix} \begin{pmatrix} S \\ T \end{pmatrix} + \begin{pmatrix} \theta \\ \phi \end{pmatrix} , \tag{24}$$

where Λ, γ, λ, and κ depend on t, and where θ and ϕ depend on $\mathbf{x}$ and t. In place of the condition $\Lambda \neq 0$, we impose that $\Delta = \kappa\Lambda - \lambda\gamma \neq 0$, so that (24) is invertible. This is the transformation group $\mathcal{G}$, modeled on $GL(2, \mathbf{R})$, with which we shall now work; the earlier gauge group is the subgroup with $\lambda \equiv 0$, $\kappa \equiv 1$, and $\phi \equiv 0$. We thus treat the phase and the logarithm of the amplitude on an equal footing. The logarithmic variables T and S are, of course, familiar from earlier hydrodynamical and stochastic versions of quantum mechanics [28, 29]; but they normally are treated quite asymmetrically.

We immediately see that Eq. (8) must be generalized further for it to be invariant under $\mathcal{G}$. This is accomplished by complexifying the coefficients and external potentials, to obtain Eq. (10)—a procedure that is natural, as Eq. (24) can be obtained by complexifying Λ, γ, and θ in the transformation from ψ to ψ'.

Since so many terms in our equations involve logarithmic derivatives, let us continue with the variables S and T. The operation of multiplying ψ by a complex scalar is then to add real constants to S and to T. The homogeneous terms in Eq. (5) become, $R_1 = \nabla^2 S + 2\nabla S \cdot \nabla T$, $R_2 = 2\nabla^2 T + 4(\nabla T)^2$, $R_3 = (\nabla S)^2$, $R_4 = 2\nabla S \cdot \nabla T$, and $R_5 = 4(\nabla T)^2$. We now write the new, general nonlinear Schrödinger equation (10) as a pair of coupled partial differential equations for the extended real-valued functions S and T, which are first order in time but have general second-order and quadratic terms:

$$\begin{aligned} \dot{S} &= a_1\nabla^2 S + a_2\nabla^2 T + a_3(\nabla S)^2 + a_4\nabla S\cdot\nabla T + a_5(\nabla T)^2 \\ &\qquad + a_6 S + a_7 T + u_0 + \mathbf{u}_1\cdot\nabla S + \mathbf{u}_2\cdot\nabla T, \\ \dot{T} &= b_1\nabla^2 S + b_2\nabla^2 T + b_3(\nabla S)^2 + b_4\nabla S\cdot\nabla T + b_5(\nabla T)^2 \\ &\qquad + b_6 S + b_7 T + v_0 + \mathbf{v}_1\cdot\nabla S + \mathbf{v}_2\cdot\nabla T. \end{aligned} \tag{25}$$

The relation between Eq. (25) and and Eq. (10) is straightforward:

$$\begin{array}{rcl} a_1 = -\mu_1\,, & & b_1 = \nu_1\,, \\ a_2 = -2\mu_2\,, & & b_2 = 2\nu_2\,, \\ a_3 = -\mu_3\,, & & b_3 = \nu_3\,, \\ a_4 = -2\mu_1 - 2\mu_4\,, & & b_4 = 2\nu_1 + 2\nu_4\,, \\ a_5 = -4\mu_2 - 4\mu_5\,, & & b_5 = 4\nu_2 + 4\nu_5\,, \\ a_6 = -\alpha_2\,, & & b_6 = \delta_2\,, \\ a_7 = -2\alpha_1\,, & & b_7 = 2\delta_1\,, \\ u_0 = -U - \nabla\cdot\mathcal{A}_1\,, & & v_0 = \mathcal{T} + \nabla\cdot\mathcal{A}\,, \\ \mathbf{u}_1 = -\mathcal{A}_2\,, & & \mathbf{v}_1 = \mathcal{D}\,, \\ \mathbf{u}_2 = -2\mathcal{A}_1\,, & & \mathbf{v}_2 = 2\mathcal{A} \end{array} . \tag{26}$$

Of course Eq. (8) is embedded in (25), as are many other interesting equations of mathematical physics. For reference, the usual, linear Schrödinger equation (7) corresponds to

$$a_1 = 0\,, \quad a_2 = \frac{\hbar}{2m}\,, \quad a_3 = ,-\frac{\hbar}{2m}\,, \quad a_4 = 0\,, \quad a_5 = \frac{\hbar}{2m}\,, \quad a_6 = a_7 = 0\,,$$

$$u_0 = -\frac{1}{\hbar}\left(V + e\Phi\right) - \frac{e^2}{2m\hbar c^2}\,\mathbf{A}^2\,, \quad \mathbf{u}_1 = \frac{e}{mc}\mathbf{A}\,, \quad \mathbf{u}_2 = 0\,,$$

$$b_1 = -\frac{\hbar}{2m}\,, \quad b_2 = 0 \quad b_3 = 0 \quad b_4 = -\frac{\hbar}{m}\,, \quad b_5 = 0\,, \quad b_6 = b_7 = 0\,,$$

$$v_0 = \frac{e}{2mc}\,\nabla\cdot\mathbf{A}, \quad \mathbf{v}_1 = 0\,, \quad \mathbf{v}_2 = \frac{e}{mc}\mathbf{A}\,. \tag{27}$$

Now the coefficients a_j, b_j obey the following transformation laws under (24), with the determinant $\Delta = \kappa\Lambda - \lambda\gamma$:

$$\begin{bmatrix} a_1' \\ a_2' \\ b_1' \\ b_2' \end{bmatrix} = \Delta^{-1} \begin{bmatrix} \kappa\Lambda & -\lambda\Lambda & \kappa\gamma & -\lambda\gamma \\ -\gamma\Lambda & \Lambda^2 & -\gamma^2 & \gamma\Lambda \\ \kappa\lambda & \lambda^2 & \kappa^2 & -\kappa\lambda \\ -\lambda\gamma & \lambda\Lambda & -\kappa\gamma & \kappa\Lambda \end{bmatrix} \begin{bmatrix} a_1 \\ a_2 \\ b_1 \\ b_2 \end{bmatrix}; \tag{28}$$

$$\begin{bmatrix} a_3' \\ a_4' \\ a_5' \\ b_3' \\ b_4' \\ b_5' \end{bmatrix} = \Delta^{-2}\,\mathcal{M} \begin{bmatrix} a_3 \\ a_4 \\ a_5 \\ b_3 \\ b_4 \\ b_5 \end{bmatrix}, \quad \text{where} \tag{29}$$

$$\mathcal{M} = \begin{bmatrix} \kappa^2\Lambda & -\kappa\lambda\Lambda & \lambda^2\Lambda & \kappa^2\gamma & -\kappa\lambda\gamma & \lambda^2\gamma \\ -2\kappa\gamma\Lambda & \Lambda(\kappa\Lambda+\lambda\gamma) & -2\lambda\Lambda^2 & -2\kappa\gamma^2 & \gamma(\kappa\Lambda+\lambda\gamma) & -2\lambda\gamma\Lambda \\ \gamma^2\Lambda & -\gamma\Lambda^2 & \Lambda^3 & \gamma^3 & -\gamma^2\Lambda & \gamma\Lambda^2 \\ \kappa^2\lambda & -\kappa\lambda^2 & \lambda^3 & \kappa^3 & -\kappa^2\lambda & \kappa\lambda^2 \\ -2\kappa\lambda\gamma & \lambda(\kappa\Lambda+\lambda\gamma) & -2\lambda^2\Lambda & -2\kappa^2\gamma & \kappa(\kappa\Lambda+\lambda\gamma) & -2\kappa\lambda\Lambda \\ \lambda\gamma^2 & -\lambda\gamma\Lambda & -\lambda\Lambda^2 & \kappa\gamma^2 & -\kappa\gamma\Lambda & \kappa\Lambda^2 \end{bmatrix};$$

and

$$\begin{bmatrix} a_6' \\ a_7' \\ b_6' \\ b_7' \end{bmatrix} = \Delta^{-1} \begin{bmatrix} \kappa\Lambda & -\lambda\Lambda & \kappa\gamma & -\lambda\gamma \\ -\gamma\Lambda & \Lambda^2 & -\gamma^2 & \gamma\Lambda \\ \kappa\lambda & \lambda^2 & \kappa^2 & -\kappa\lambda \\ -\lambda\gamma & \lambda\Lambda & -\kappa\gamma & \kappa\Lambda \end{bmatrix} \begin{bmatrix} a_6 \\ a_7 \\ b_6 \\ b_7 \end{bmatrix} + \Delta^{-1} \begin{bmatrix} \kappa\dot{\Lambda} - \lambda\dot{\gamma} \\ \Lambda\dot{\gamma} - \gamma\dot{\Lambda} \\ \kappa\dot{\lambda} - \lambda\dot{\kappa} \\ \Lambda\dot{\kappa} - \gamma\dot{\lambda} \end{bmatrix}. \tag{30}$$

The behavior of the external fields under generalized gauge transformation is more complicated. The transformed vector fields $\mathbf{u}_1'$, $\mathbf{u}_2'$, $\mathbf{v}_1'$, and $\mathbf{v}_2'$ are linear combinations of the six coefficients a_3, a_4, a_5, b_3, b_4, b_5 and the four vector fields $\mathbf{u}_1$, $\mathbf{u}_2$, $\mathbf{v}_1$, and $\mathbf{v}_2$; for example, the matrix element of $\mathbf{u}_1'$ by a_3 is $\Delta^{-2}\,(-2\kappa^2\Lambda\nabla\theta + 2\kappa\gamma\Lambda\nabla\phi)$, and its matrix element by $\mathbf{v}_2$ is $\Delta^{-1}\,(-\lambda\gamma)$. The transformed scalar fields u_0' and v_0' are linear combinations of all fourteen coefficients $a_1 \ldots a_7$ and $b_1 \ldots b_7$, the scalar fields u_0 and v_0, and the four vector fields, plus affine terms that depend on the time-derivatives of Λ γ, λ, κ, θ, and ϕ. Probably little insight would be added by reproducing all the equations here.

Now we come to the main point. The generalization that is proposed will work (i.e., allow a gauge-invariant theory of measurement) only if it is possible to write combinations formed from S and T that are invariant under Eq. (24)—just as the earlier combinations $\rho = \exp[2T]$ and $\mathbf{J}^{\mathrm{gi}}/\rho = -2\nu_1\nabla S - 4\nu_2\nabla T - 2\mathcal{A}$ are invariant under the smaller group. Consider for simplicity only the matrix part of (24); that is, set $\theta = \phi = 0$; call the gauge transformation matrix A. Suppose that d_1, d_2 are some coefficients depending on the a_j and the b_j. Then d_1S+d_2T is invariant under A if and only if $[d_1\ d_2]\,A^{-1} = [d_1'\ d_2']$. From (29), we observe that the choice $d_1 = 2a_3+b_4$ and $d_2 = a_4+2b_5$ obeys this condition. Hence d_1S+d_2T can serve as one of the desired invariant combinations. Next let $L_1 = a_1S + a_2T$ and $L_2 = b_1S + b_2T$. Then the pair (L_1, L_2) transforms under A exactly as does the pair (S, T), whence $d_1L_1 + d_2L_2$ is also an invariant. In fact, any combination $d_1(\sigma L_1 + \tau S) + d_2(\sigma L_2 + \tau T)$, where σ and τ are fully invariant combination of the coefficients, will be invariant; and, of course, any function of invariants is invariant. It is straightforward to verify that $a_1 + b_2 = 2\tau_1$ and $a_1b_2 - a_2b_1 = 2\tau_2$, which were earlier identified as gauge invariants for (11), are also invariants under (24). We shall interpret $\tau_2 > 0$ as characterizing the class of Eqs. (25) that pertain to quantum mechanics, with $\tau_2 \to 0$ defining the classical limit in a gauge-independent way.

To conclude, the desired invariant combinations of S and T exist. There is enough flexibility to permit a choice that reduces to the usual formulas in the case of the linear

Schrödinger equation. In this way we can construct a positive definite, gauge-invariant probability density $\mathcal{P}^{\mathrm{gi}}$ and gauge-invariant current $\mathcal{J}^{\mathrm{gi}}$. A large subfamily of Eqs. (10) have solutions for which $\mathcal{P}^{\mathrm{gi}}$ and $\mathcal{J}^{\mathrm{gi}}$ obey the desired continuity equation, so that the total probability is conserved. And it is important to stress that a (smaller) subclass of Eqs. (10) is equivalent to ordinary quantum mechanics by way of generalized nonlinear gauge transformations, so that we are assured the new formalism is consistent. We can even exchange S and $\ln R$ in ordinary quantum mechanics, by taking $\gamma = \lambda = 1$, $\kappa = \Lambda = 0$.

It is clear that in this wider framework, many of the tacit assumptions of quantum mechanics no longer hold. For instance, integrability of the probability density function is only equivalent to square integrability of the wave function in certain gauges, so that we are often outside the usual Hilbert space of quantum mechanics.

Further details of these results will be presented elsewhere.

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